James Stewart

Essential Calculus

EARLY TRANSCENDENTALS

Second Edition

James Stewart

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Second Edition

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ESSENTIAL CALCULUS

Early Transcendentals

SECOND EDITION

JAMES STEWART
McMaster University
and
University of Toronto

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James Stewart

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PREFACE

This book is a response to those instructors who feel that calculus textbooks are too big. In writing the book I asked myself: What is *essential* for a three-semester calculus course for scientists and engineers?

The book is about two-thirds the size of my other calculus books (*Calculus,* Seventh Edition and *Calculus, Early Transcendentals,* Seventh Edition) and yet it contains almost all of the same topics. I have achieved relative brevity mainly by condensing the exposition and by putting some of the features on the website **www.stewartcalculus.com**. Here, in more detail are some of the ways I have reduced the bulk:

- I have organized topics in an efficient way and rewritten some sections with briefer exposition.
- The design saves space. In particular, chapter opening spreads and photographs have been eliminated.
- The number of examples is slightly reduced. Additional examples are provided online.
- The number of exercises is somewhat reduced, though most instructors will find that there are plenty. In addition, instructors have access to the archived problems on the website.
- Although I think projects can be a very valuable experience for students, I have removed them from the book and placed them on the website.
- A discussion of the principles of problem solving and a collection of challenging problems for each chapter have been moved to the website.

Despite the reduced size of the book, there is still a modern flavor: Conceptual understanding and technology are not neglected, though they are not as prominent as in my other books.

ALTERNATE VERSIONS

I have written several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Essential Calculus*, Second Edition, is similar to the present textbook except that the logarithm is defined as an integral and so the exponential, logarithmic, and inverse trigonometric functions are covered later than in the present book.
- *Calculus: Early Transcendentals*, Seventh Edition, has more complete coverage of calculus than the present book, with somewhat more examples and exercises.
- *Calculus: Early Transcendentals*, Seventh Edition, Hybrid Version, is similar to *Calculus: Early Transcendentals*, Seventh Edition, in content and coverage except that all of the end-of-section exercises are available only in Enhanced WebAssign. The printed text includes all end-of-chapter review material.
- *Calculus*, Seventh Edition, is similar to *Calculus: Early Transcendentals*, Seventh Edition, except that the exponential, logarithmic, and inverse trigonometric functions are covered in the second semester. It is also available in a Hybrid Version.

- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters. It is also available in a Hybrid Version.
- *Calculus: Early Vectors* introduces vectors and vector functions in the first semester and integrates them throughout the book. It is suitable for students taking Engineering and Physics courses concurrently with calculus.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences. It is also available in a Hybrid Version.

WHAT'S NEW IN THE SECOND EDITION?

The changes have resulted from talking with my colleagues and students at the University of Toronto and from reading journals, as well as suggestions from users and reviewers. Here are some of the many improvements that I've incorporated into this edition:

- At the beginning of the book there are four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry. Answers are given and students who don't do well are referred to where they should seek help (Appendixes, review sections of Chapter 1, and the website).
- Section 7.5 (Area of a Surface of Revolution) is new. I had asked reviewers if there was any topic missing from the first edition that they regarded as essential. This was the only topic that was mentioned by more than one reviewer.
- Some material has been rewritten for greater clarity or for better motivation. See, for instance, the introduction to maximum and minimum values on pages 203–04 and the introduction to series on page 436.
- New examples have been added (see Example 4 on page 725 for instance). And the solutions to some of the existing examples have been amplified. A case in point: I added details to the solution of Example 1.4.9 because when I taught Section 1.4 from the first edition I realized that students need more guidance when setting up inequalities for the Squeeze Theorem.
- The data in examples and exercises have been updated to be more timely.
- Several new historical margin notes have been added.
- About 40% of the exercises are new. Here are some of my favorites: 1.6.43, 2.2.13–14, 2.5.59, 2.6.39–40, 3.2.70, 4.3.66, 5.3.44–45, 7.6.24, 8.2.29–30, 8.7.67–68, 10.1.38, 10.4.43–44
- The animations in *Tools for Enriching Calculus* (TEC) have been completely redesigned and are accessible in Enhanced WebAssign, CourseMate, and PowerLecture. Selected Visuals and Modules are available at www.stewartcalculus.com.

CONTENT

DIAGNOSTIC TESTS ▪ The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

CHAPTER 1 ▪ FUNCTIONS AND LIMITS After a brief review of the basic functions, limits and continuity are introduced, including limits of trigonometric functions, limits involving infinity, and precise definitions.

CHAPTER 2 ▪ DERIVATIVES The material on derivatives is covered in two sections in order to give students time to get used to the idea of a derivative as a function. The formulas for the derivatives of the sine and cosine functions are derived in the section on basic differentiation formulas. Exercises explore the meanings of derivatives in various contexts.

CHAPTER 3 ▪ INVERSE FUNCTIONS: EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS Exponential functions are defined first and the number e is defined as a limit. Logarithms are then defined as inverse functions. Applications to exponential growth and decay follow. Inverse trigonometric functions and hyperbolic functions are also covered here. L'Hospital's Rule is included in this chapter because limits of transcendental functions so often require it.

CHAPTER 4 ▪ APPLICATIONS OF DIFFERENTIATION The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. The section on curve sketching includes a brief treatment of graphing with technology. The section on optimization problems contains a brief discussion of applications to business and economics.

CHAPTER 5 ▪ INTEGRALS The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix B.) A quite general definition of the definite integral (with unequal subintervals) is given initially before regular partitions are employed. Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.

CHAPTER 6 ▪ TECHNIQUES OF INTEGRATION All the standard methods are covered, as well as computer algebra systems, numerical methods, and improper integrals.

CHAPTER 7 ▪ APPLICATIONS OF INTEGRATION General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral. The chapter concludes with an introduction to differential equations, including separable equations and direction fields.

CHAPTER 8 ▪ SERIES The convergence tests have intuitive justifications as well as formal proofs. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those based on Taylor's Formula (with Lagrange's form of the remainder term) and those from graphing devices.

CHAPTER 9 ▪ PARAMETRIC EQUATIONS AND POLAR COORDINATES This chapter introduces parametric and polar curves and applies the methods of calculus to them. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 10.

CHAPTER 10 ▪ VECTORS AND THE GEOMETRY OF SPACE In addition to the material on vectors, dot and cross products, lines, planes, and surfaces, this chapter covers vector-valued functions, length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

CHAPTER 11 ▪ PARTIAL DERIVATIVES In view of the fact that many students have difficulty forming mental pictures of the concepts of this chapter, I've placed a special emphasis on graphics to elucidate such ideas as graphs, contour maps, directional derivatives, gradients, and Lagrange multipliers.

CHAPTER 12 ▪ MULTIPLE INTEGRALS Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals.

CHAPTER 13 ▪ VECTOR CALCULUS The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

WEBSITE

The web site **www.stewartcalulus.com** includes the following.

- Review of Algebra, Trigonometry, Analytic Geometry, and Conic Sections
- Homework Hints
- Additional Examples
- Projects
- Archived Problems (drill exercises that were in previous editions of my other books), together with their solutions
- Challenge Problems
- Lies My Calculator and Computer Told Me
- Additional Topics (complete with exercise sets): Principles of Problem Solving, Strategy for Integration, Strategy for Testing Series, Fourier Series, Linear Differential Equations, Second Order Linear Differential Equations, Nonhomogeneous Linear Equations, Applications of Second Order Differential Equations, Using Series to Solve Differential Equations, Complex Numbers, Rotation of Axes
- Links, for particular topics, to outside Web resources
- History of Mathematics, with links to the better historical websites
- TEC animations for Chapters 2 and 5

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The idea for this book came from my former editor Bob Pirtle, who had been hearing of the desire for a much shorter calculus text from numerous instructors. I thank my present editor Liz Covello for sustaining and supporting this idea in the second edition.

JAMES STEWART

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By James Stewart, Harvey Keynes, Dan Clegg, and developer Hu Hohn

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By Dennis Ebersole, Doris Schattschneider, Alicia Sevilla, and Kay Somers

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Linear Algebra for Calculus

By Konrad J. Heuvers, William P. Francis, John H. Kuisti, Deborah F. Lockhart, Daniel S. Moak, and Gene M. Ortner

ISBN 0-534-25248-6

This comprehensive book, designed to supplement the calculus course, provides an introduction to and review of the basic ideas of linear algebra.

TO THE STUDENT

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, I suggest that you cover up the solution and try solving the problem yourself. You'll get a lot more from looking at the solution if you do so.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix E. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/\left(1 + \sqrt{2}\right)$, then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. (The use of these graphing devices and some of the pitfalls that you may encounter are discussed on **stewartcalculus.com**. Go to *Additional Topics* and click on *Graphing Calculators and Computers*.) But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol CAS is reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-89/92) are required.

You will also encounter the symbol ⊘, which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

Tools for Enriching Calculus, which is a companion to this text, is referred to by means of the symbol TEC and can be accessed in Enhanced WebAssign and CourseMate (selected Visuals and Modules are available at www.stewartcalculus.com). It directs you to modules in which you can explore aspects of calculus for which the computer is particularly useful.

There is a lot of useful information on the website **stewartcalculus.com**. There you will find a review of precalculus topics (in case your algebraic skills are rusty), as well as *Homework Hints* (see the following paragraph), *Additional Examples* (see below), *Challenge Problems*, *Projects*, *Lies My Calculator and Computer Told Me*, (explaining why calculators sometimes give the wrong answer), *Additional Topics*, and links to outside resources.

Homework Hints for representative exercises are indicated by printing the exercise number in blue: **5.** These hints can be found on stewartcalculus.com as well as Enhanced WebAssign and CourseMate. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. You need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

You will see margin notes in some sections directing you to *Additional Examples* on the website. You will also see the symbol V beside two or three of the examples in every section of the text. This means that there are videos (in Enhanced WebAssign and CourseMate) of instructors explaining those examples in greater detail.

I recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.

JAMES STEWART

DIAGNOSTIC TESTS

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A DIAGNOSTIC TEST: ALGEBRA

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$

(b) $(3a^3b^3)(4ab^2)^2$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5} - 2}$ (b) $\dfrac{\sqrt{4 + h} - 2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leqslant 17$ (b) $x^2 < 2x + 8$

(c) $x(x - 1)(x + 2) > 0$ (d) $|x - 4| < 3$

(e) $\dfrac{2x - 3}{x + 1} \leqslant 1$

10. State whether each equation is true or false.

(a) $(p + q)^2 = p^2 + q^2$ (b) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$ (d) $\dfrac{1 + TC}{C} = 1 + T$

(e) $\dfrac{1}{x - y} = \dfrac{1}{x} - \dfrac{1}{y}$ (f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a - b}$

A ANSWERS TO DIAGNOSTIC TEST A: ALGEBRA

1. (a) 81 (b) -81 (c) $\frac{1}{81}$
(d) 25 (e) $\frac{9}{4}$ (f) $\frac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x - 5)(2x + 5)$ (b) $(2x - 3)(x + 4)$
(c) $(x - 3)(x - 2)(x + 2)$ (d) $x(x + 3)(x^2 - 3x + 9)$
(e) $3x^{-1/2}(x - 1)(x - 2)$ (f) $xy(x - 2)(x + 2)$

5. (a) $\dfrac{x + 2}{x - 2}$ (b) $\dfrac{x - 1}{x - 3}$
(c) $\dfrac{1}{x - 2}$ (d) $-(x + y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4 + h} + 2}$

7. (a) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ (b) $2(x - 3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\frac{2}{3}, \frac{22}{3}$
(g) $\frac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4]$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you have had difficulty with these problems, you may wish to consult the Review of Algebra on the website www.stewartcalculus.com.

B DIAGNOSTIC TEST: ANALYTIC GEOMETRY

1. Find an equation for the line that passes through the point $(2, -5)$ and
(a) has slope -3
(b) is parallel to the x-axis
(c) is parallel to the y-axis
(d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
(a) Find the slope of the line that contains A and B.
(b) Find an equation of the line that passes through A and B. What are the intercepts?
(c) Find the midpoint of the segment AB.
(d) Find the length of the segment AB.
(e) Find an equation of the perpendicular bisector of AB.
(f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.
(a) $-1 \leqslant y \leqslant 3$
(b) $|x| < 4$ and $|y| < 2$
(c) $y < 1 - \frac{1}{2}x$
(d) $y \geqslant x^2 - 1$
(e) $x^2 + y^2 < 4$
(f) $9x^2 + 16y^2 = 144$

B ANSWERS TO DIAGNOSTIC TEST B: ANALYTIC GEOMETRY

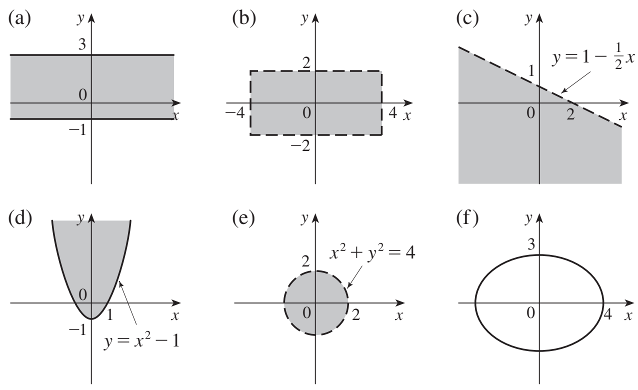
1. (a) $y = -3x + 1$ (b) $y = -5$
(c) $x = 2$ (d) $y = \frac{1}{2}x - 6$

2. $(x + 1)^2 + (y - 4)^2 = 52$

3. Center $(3, -5)$, radius 5

4. (a) $-\frac{4}{3}$
(b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
(c) $(-1, -4)$
(d) 20
(e) $3x - 4y = 13$
(f) $(x + 1)^2 + (y + 4)^2 = 100$

5.



If you have had difficulty with these problems, you may wish to consult the review of analytic geometry on the website www.stewartcalculus.com.

C DIAGNOSTIC TEST: FUNCTIONS

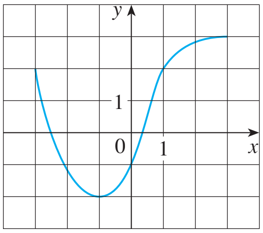


FIGURE FOR PROBLEM 1

1. The graph of a function f is given at the left.
(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.

(a) $f(x) = \dfrac{2x+1}{x^2 + x - 2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2 + 1}$ (c) $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

4. How are graphs of the functions obtained from the graph of f?
(a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x - 3) + 2$

5. Without using a calculator, make a rough sketch of the graph.
(a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = (x - 2)^3 + 3$
(d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
(g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
(a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

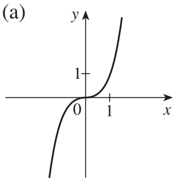
C | ANSWERS TO DIAGNOSTIC TEST C: FUNCTIONS

1. (a) -2 (b) 2.8
(c) $-3, 1$ (d) $-2.5, 0.3$
(e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
(b) $(-\infty, \infty)$
(c) $(-\infty, -1] \cup [1, 4]$

4. (a) Reflect about the x-axis
(b) Stretch vertically by a factor of 2, then shift 1 unit downward
(c) Shift 3 units to the right and 2 units upward

5. (a)


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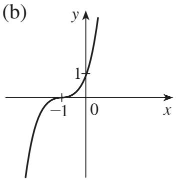

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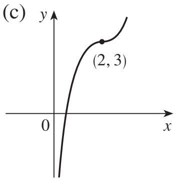

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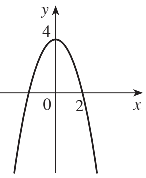

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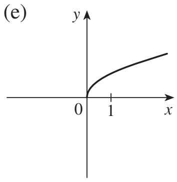
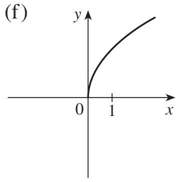
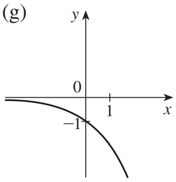
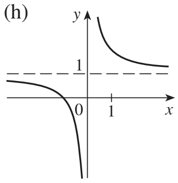
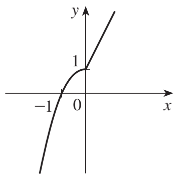


(f)


(g)


(h)


6. (a) $-3, 3$
(b)


7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$
(b) $(g \circ f)(x) = 2x^2 + 4x - 5$
(c) $(g \circ g \circ g)(x) = 8x - 21$

If you have had difficulty with these problems, you should look at Sections 1.1–1.2 of this book.

D DIAGNOSTIC TEST: TRIGONOMETRY

1. Convert from degrees to radians.
(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.
(a) $5\pi/6$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of $30°$.

4. Find the exact values.
(a) $\tan(\pi/3)$ (b) $\sin(7\pi/6)$ (c) $\sec(5\pi/3)$

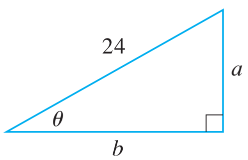


FIGURE FOR PROBLEM 5

5. Express the lengths a and b in the figure in terms of θ.

6. If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.

7. Prove the identities.
(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$
(b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \leqslant x \leqslant 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

D ANSWERS TO DIAGNOSTIC TEST D: TRIGONOMETRY

1. (a) $5\pi/3$ (b) $-\pi/10$

2. (a) $150°$ (b) $(360/\pi)° \approx 114.6°$

3. 2π cm

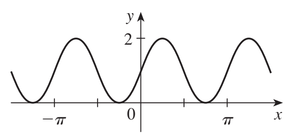
4. (a) $\sqrt{3}$ (b) $-\frac{1}{2}$ (c) 2

5. (a) $24\sin\theta$ (b) $24\cos\theta$

6. $\frac{1}{15}(4 + 6\sqrt{2})$

8. $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



If you have had difficulty with these problems, you should look at Appendix A of this book.

1 FUNCTIONS AND LIMITS

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. So in this first chapter we begin our study of calculus by investigating how the values of functions change and approach limits.

1.1 FUNCTIONS AND THEIR REPRESENTATIONS

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

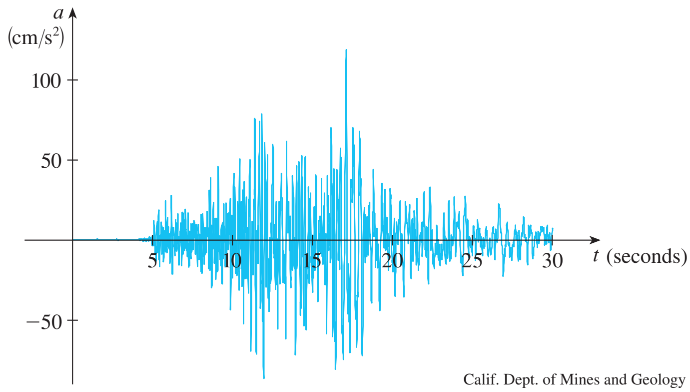


FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

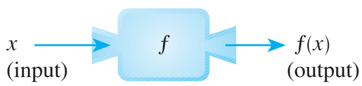


FIGURE 2
Machine diagram for a function f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

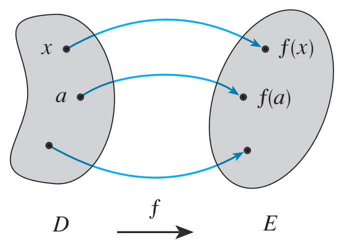


FIGURE 3
Arrow diagram for f

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

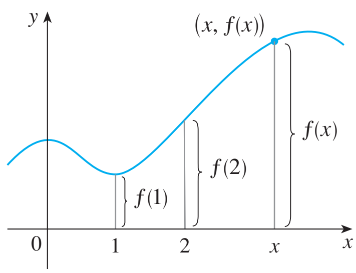


FIGURE 4

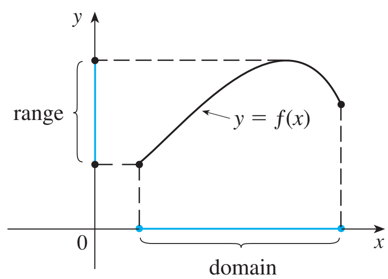


FIGURE 5

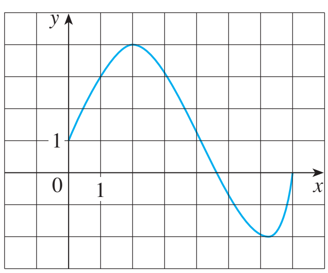


FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

▪ The notation for intervals is given on Reference Page 3. The Reference Pages are located at the back of the book.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$

REPRESENTATIONS OF FUNCTIONS

▪ **www.stewartcalculus.com**
See Additional Examples A, B.

There are four possible ways to represent a function:

- verbally (by a description in words)
- visually (by a graph)
- numerically (by a table of values)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it is often useful to go from one representation to another to gain additional insight into the function. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 7. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, we could use a graphing calculator with exponential regression capabilities to obtain the approximation

t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 8 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

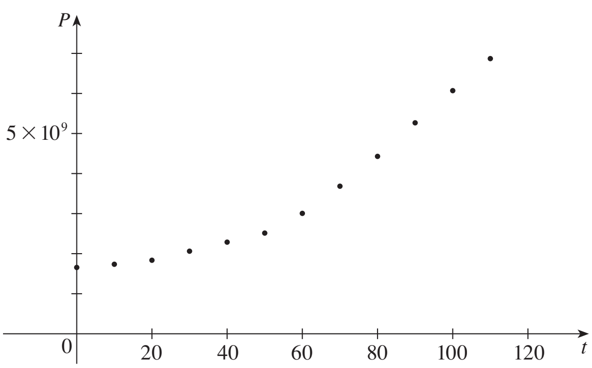


FIGURE 7 Scatter plot of data points for population growth

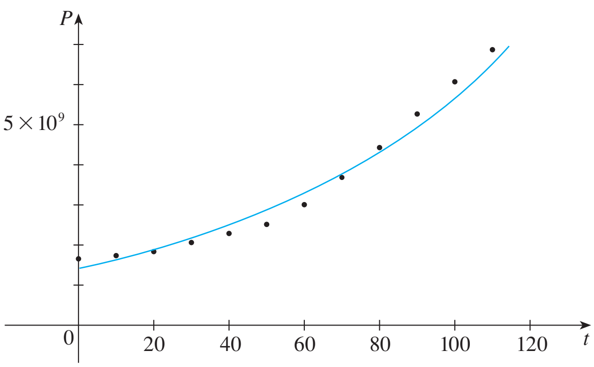


FIGURE 8 Graph of a mathematical model for population growth

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

▪ A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leqslant 1$	0.88
$1 < w \leqslant 2$	1.08
$2 < w \leqslant 3$	1.28
$3 < w \leqslant 4$	1.48
$4 < w \leqslant 5$	1.68
⋮	⋮

C. Again the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2011 is as follows: The cost is 88 cents for up to one ounce, plus 20 cents for each successive ounce (or less) up to 13 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 6).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 2 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

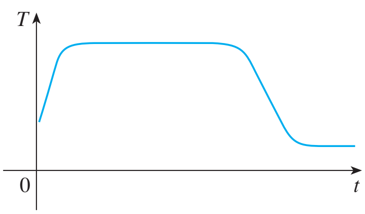


FIGURE 9

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 9. ■

EXAMPLE 3 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

▪ If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since
$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$
and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is $\{x \mid x \neq 0, x \neq 1\}$, which could also be written in interval notation as $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. ■

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 10. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

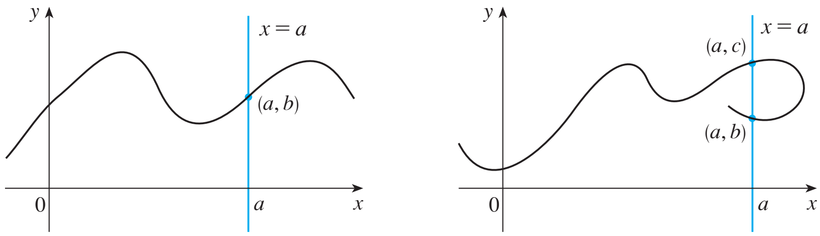


FIGURE 10

PIECEWISE DEFINED FUNCTIONS

The functions in the following three examples are defined by different formulas in different parts of their domains.

V EXAMPLE 4 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant 1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > 1$, then the value of $f(x)$ is x^2.

$$\text{Since } 0 \leqslant 1, \text{ we have } f(0) = 1 - 0 = 1.$$

$$\text{Since } 1 \leqslant 1, \text{ we have } f(1) = 1 - 1 = 0.$$

$$\text{Since } 2 > 1, \text{ we have } f(2) = 2^2 = 4.$$

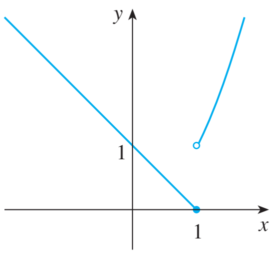


FIGURE 11

How do we draw the graph of f? We observe that if $x \leqslant 1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = 1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = 1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 11. The solid dot indicates that the point $(1, 0)$ is included on the graph; the open dot indicates that the point $(1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

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For a more extensive review of absolute values, click on *Review of Algebra*.

In general, we have

$$\boxed{\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \qquad \text{if } a < 0 \end{aligned}}$$

(Remember that if a is negative, then $-a$ is positive.)

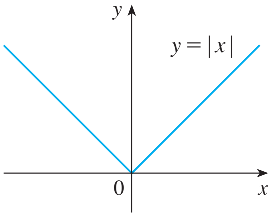

FIGURE 12

EXAMPLE 5 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 4, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 12). ■

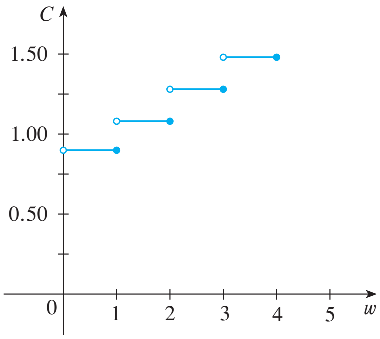

FIGURE 13

EXAMPLE 6 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from the table of values on page 4, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \leqslant 1 \\ 1.08 & \text{if } 1 < w \leqslant 2 \\ 1.28 & \text{if } 2 < w \leqslant 3 \\ 1.48 & \text{if } 3 < w \leqslant 4 \end{cases}$$

The graph is shown in Figure 13. You can see why functions similar to this one are called **step functions**—they jump from one value to the next. ■

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See Additional Examples C, D.

SYMMETRY

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 14). This means that if we have plotted the graph of

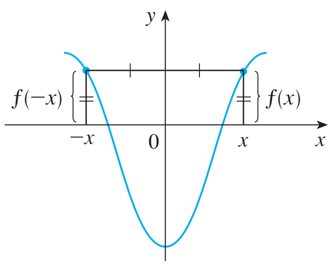


FIGURE 14 An even function

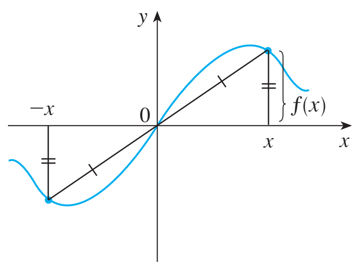


FIGURE 15 An odd function

f for $x \geqslant 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = (-1)^3 x^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 15 on page 6). If we already have the graph of f for $x \geqslant 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

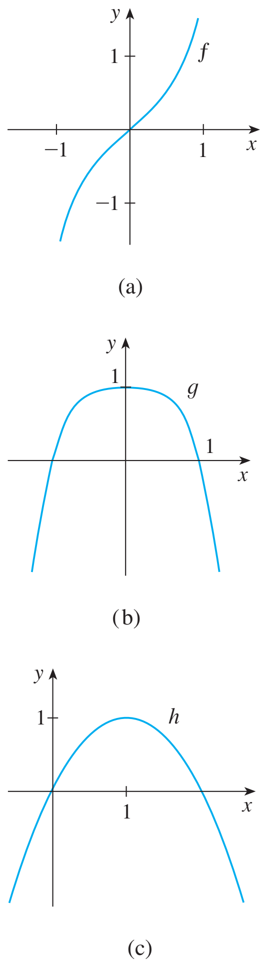


FIGURE 16

V EXAMPLE 7 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 7 are shown in Figure 16. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

INCREASING AND DECREASING FUNCTIONS

The graph shown in Figure 17 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

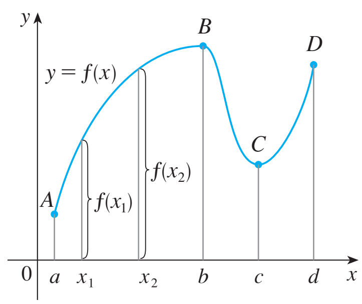


FIGURE 17

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 18 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

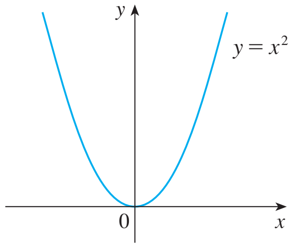


FIGURE 18

1.1 EXERCISES

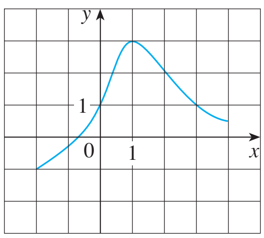
1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

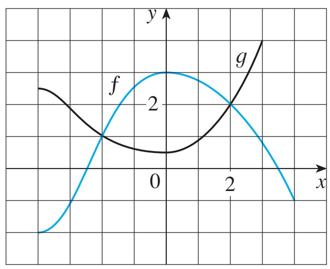
$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

is it true that $f = g$?

3. The graph of a function f is given.
(a) State the value of $f(1)$.
(b) Estimate the value of $f(-1)$.
(c) For what values of x is $f(x) = 1$?
(d) Estimate the value of x such that $f(x) = 0$.
(e) State the domain and range of f.
(f) On what interval is f increasing?

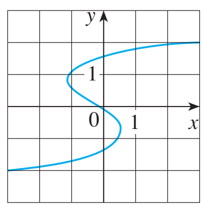


4. The graphs of f and g are given.
(a) State the values of $f(-4)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) Estimate the solution of the equation $f(x) = -1$.
(d) On what interval is f decreasing?
(e) State the domain and range of f.
(f) State the domain and range of g.



5–8 ■ Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

5.



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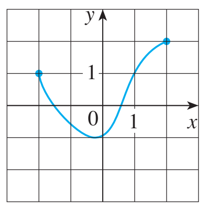

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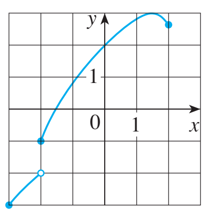

8.